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Locally Small Riemann Sum Property of a Vector Valued Function in Locally Compact Metric Space

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Abstract

In this paper, we defined a locally small Riemann sum (LSRS) property relative to volume function ϑ of vector-valued function on a cell E in a nondiscrete locally compact metric space X. By the definition, if a function f has LSRS property relative to ϑ on cell E is denoted by $f \in LSRS(E,\vartheta)$, then it can be proved that: (i) if $f, g \in LSRS(E,\vartheta)$, then $f + g \in LSRS(E,\vartheta)$ and $\alpha f \in LSRS(E,\vartheta)$ for any real number α; (ii) if $f \in LSRS(E,\vartheta)$, then f is bounded on E; (iii) if $f \in LSRS(E,\vartheta)$, then f is Henstock integrable relative to ϑ on E; (iv) if $(Y, \|.\|)$ is a complete normed space and $f: E \to Y$ is Henstock integrable relative to ϑ, then $f \in LSRS(E,\vartheta)$.

Keywords: cell, locally compact metric space, volume function, Henstock integrable, locally small Riemann sum

1 Introduction

Research on Locally small Riemann sum (LSRS) property of real-valued function defined on a cell $E \subset \mathbb{R}$ has been conducted by some mathematicians ([1], [2], and [9]). The result for this research is to prove an equivalence to function having LSRS property on cell $[a,b]$ and Henstock integrable function on cell$[a,b]$, in terms of f has the LSRS property on $[a,b]$ if and only if f is Henstock integrable on cell $[a,b]$.

Based on the research conducted by mentioned Mathematicians above and referred to the result of latest research: Henstock integrable of vector-valued function in locally compact metric space ([5]) and Measurable vector-valued function relative to volume function ϑ on a cell in nondiscrete locally compact metric space ([7]), will be defined LSRS property of vector-valued function in nondiscrete locally compact metric space and next is to investigate the related properties.

The differences of this research related to the previous research conducted by another scholar can be categorized in three differentaspect which are: measurement used, domain of the function, and value of the function. In the previous research, the measurement used in the analysis is Lebesgue measure. While in this research, we generate Lebesgue measure by measuring μ generated by volume function ϑ. The domain of the function is also changed from cell $[a,b] \subset \mathbb{R}$ into an abstraction of cell in $\mathbb{R}$ which is cell E in nondiscrete locally compact metric space X. The third difference is about the value of the function which is real (in usual metric space $\mathbb{R}$) in the previous study and become a vector in normed space $(Y, \|.\|)$.

It is necessary to know that metric space $\mathbb{R}$ is a complete metric space, in terms of for every Cauchy sequence in usual metric space $\mathbb{R}$ is convergent, while normed space $(Y, \|.\|)$ is not necessarily a complete metric space ([8], p. 43, 52). Based on these facts, we will find a necessary condition to make a Henstock integrable of vector-valued function relative to volume function ϑ on a cell E in a nondiscrete locally compact metric space has the LSRS property on E relative to the same volume function.

2 Basic Concept

2.1 System of Interval

A nonvoid collection $\mathcal{S} \subset 2^X$ is called a **system of intervals** if it satisfies:

(i) for every $p \in X, \{p\} \in \mathcal{S}$,
(ii) for every $p \in X$, if $cl(N(p,r))$ compact then $N(p,r) \in \mathcal{S}$,if $A \in \mathcal{S}$, then A convex, $cl(A)$ compact, $cl(A) \in \mathcal{S}$, and $int(A) \in \mathcal{S}$,
(iii) for every $A, B \in \mathcal{S}$, $A \cap B \in \mathcal{S}$,
(iv) for every A, $B \in \mathcal{S}$ there exists a collection of nonoverlapping sets $C_i \in \mathcal{S}$, $1 \leq i \leq n$ such that

$$A - B = \bigcup_{i=1}^{n} C_i.$$

Every an element of $\mathcal{S}$ is called an **interval**. Interval $A \in \mathcal{S}$ is said to be **degenerate** if $int(A) = \emptyset$ and **nondegenerate** if $int(A) \neq \emptyset$. A **cell** is a nondegenerate compact interval. A set $A \subset X$ is called **elementary set** if A is a finite union of intervals. The collection of all elementary sets in X, is denoted by $\mathcal{E}(X)$. ([3], p. 11-12).

2.2 Measure and Measureable Set

A volume function $\vartheta: \mathcal{E}(X) \to \mathbb{R}$ is called a **measure** on $\mathcal{E}(X)$ if:

(i) $\vartheta(A) = 0$ if $int(A) = \emptyset$ and $\vartheta(A) > 0$ if $int(A) \neq \emptyset$

(ii) $\vartheta(A) \leq \vartheta(B)$ for every $A, B \in \mathcal{E}(X)$ with $A \subset B$, and

$$\vartheta\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \vartheta(A_i)$$

for every nonoverlapping collection sets $\{A_i\} \subset \mathcal{E}(X)$ with $\bigcup_{i\in\mathbb{N}} A_i \in E(X)$. Since X is a locally compact metric space, then for every interval $A \in \mathcal{S}$, we have $\vartheta(A) = \vartheta(cl(A))$ and for every $x \in X$, we have $\vartheta(\{x\}) = 0$. Furthermore, if

$$\mu^*(A) = inf\left\{\sum \vartheta(A_i)\right\}$$

for all collection of open interval $\{A_i: A \subset \bigcup A_i\}$, then μ^* is an **outer measure** on X ([3], p. 20-21).

A set $E \in X$ is said to be $\boldsymbol{\mu^*}$ **−measurable** if for every $A \in X$ we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C).$$

For every interval in X is μ^* −measurable set, and a collection of all μ^* −measurable sets in X is an algebra$-\sigma$ on X. Furthermore, if

$$\{E_i: i = 1,2, \dots, n\}$$

is a collection of finite disjoint μ^* −measurable sets, then for $A \subset X$,

$$\mu^*[\cup_{i=1}^{n}(A \cap E_i)] = \sum_{i=1}^{n} \mu^*(A \cap E_I).$$

([3], p.22-23).

2.3 Henstock Integrable Function

Let X be a locally compact nondiscrete metric space, ϑ be a volume function on $E(X)$. If E is a cell in X and $\delta: E \to \mathbb{R}^+$, then a finite collection of cell-point pairs

$$P = \{(A_{x_i}, x_i), 1 \leq i \leq n\} = \{(A_x, x)\}$$

is called a **Perron δ-*fine* partition on** E if $x_i \in A_{x_i} \subset N(x_i, \delta(x_i))$ and $\{A_{x_i}\}$ is a partition on E. Furthermore, if $\{A_{x_i}\}$ is a partition of E, then P is called a **Perron δ-*fine* partition of** E.

If $E \subset X$ is a cell and $\delta: E \to \mathbb{R}^+$, then there exists a Perron δ-*fine* partition on E ([3], p. 18).

Let $(Y, \|.\|)$ be a norm space. A function $g: X \to Y$ is said to be Henstock ϑ- integrable on a cell $E \subset X$ if there exists a vector $a \in Y$ such that for any real number $\varepsilon > 0$ thre exists a function $\delta: E \to \mathbb{R}^+$ such that for any Perron δ-fine partition

$$P = \{(A_{x_i}, x_i), 1 \le i \le n\} = \{(A_x, x)\}$$

on E we have

$$\left\| \sum_{i=1}^{n} g(x_i)\, \vartheta(A_{x_i}) - a \right\| = \| P \sum g(x)\ \vartheta(A_x) - a \| < \varepsilon.$$

If $A \in \mathcal{S}$ with $\vartheta(A) = 0$, then integral- ϑ Henstock of a function $g: X \to Y$ on A is defined by null vector (θ). The set of all Henstock ϑ −integrable functions on a cell E is denoted by $H(E, \vartheta)$ and a vector a is called Henstock ϑ −integral value of a function g on E and denoted by

$$(H - \vartheta) \int g = a.$$

([5], p. 20-21).

In the following are the properties of the Henstock ϑ −integrable function used in this paper.

Theorem 2.3.1. ([5], p. 24) *If* $\boldsymbol{g} \in \boldsymbol{H}(\boldsymbol{E}, \boldsymbol{\vartheta})$, *then for every a cell* $\boldsymbol{F} \subset \boldsymbol{E}$, $\boldsymbol{g} \in \boldsymbol{H}(\boldsymbol{F}, \boldsymbol{\vartheta})$.

Theorem 2.3.2. ([4], p. 22-26) *A function* $\boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Y}$ *is Henstock* $\boldsymbol{\vartheta}$*- integrable on a cell* $\boldsymbol{E} \subset \boldsymbol{X}$ *if only if there exists an additive function* $\boldsymbol{G}: \boldsymbol{I}(\boldsymbol{E}) \to \boldsymbol{Y}$ *with property for any real number* $\varepsilon > 0$ *there exists a function* $\delta: \boldsymbol{E} \to \mathbb{R}^+$ *such that for any Perron* δ*−fine partition*

$$P = \{(A_{x_i}, x_i), 1 \le i \le n\} = \{(A_x, x)\}$$

on E*, we have*

$$\left\| \sum_{i=1}^{n} g(x_i)\, \vartheta(A_{x_i}) - G(E) \right\| = \| P \sum g(x)\ \vartheta(A_x) - G(E) \| < \varepsilon.$$

If E is a cell and $I(E)$ is a collection of all subintervals in E, then an additive function $G: I(E) \to Y$ is called Henstock ϑ −primitive of g on E if $G(\emptyset) = \boldsymbol{\theta}$ (null vector) and for every $A \in I(E)$, $A \neq \emptyset$, we have

$$G(A) = (H - \vartheta) \int_{Cl(A)} g.$$

Theorem 2.3.3. Henstock's Lemma ([4], p. 22-26) *If* $g \in H(E, \vartheta)$ *with* Henstock ϑ −primitive'*s is* G, *then for any real number* $\varepsilon > 0$ *there exists a function* $\delta: E \to \mathbb{R}^+$ *shuch that for any Perron* δ*-fine partition* $P = \{(A_x, x)\}$ on E *and any* $Q \subset P$ *we have*

$$\| Q \sum [g(x) \vartheta(A_x) - G(A_x)] \| < 2\varepsilon.$$

By this theorem, it is trivial to prove the following theorem.

Theorem 2.3.4. ([6], p. M69-M76) *A function* $g: X \to Y$ *is a Henstock* ϑ *−integrable on a cell* $E \subset X$ *if only if there is an additive function* $G: I(E) \to Y$ *with property: for any real number* $\varepsilon > 0$ *there exists a function* $\delta: E \to \mathbb{R}^+$ *shuch that for any Perron δ-fine partition* $P = \{(A_x, x)\}$ *we have*

$$P \sum \|g(x)\vartheta(A_x) - G(A_x)\| < \varepsilon.$$

Theorem 2.3.5. ([6], p.M69-M76) *If* $g \in H(E, \vartheta)$ *with* Henstock ϑ −primitive's *is* G*, then for every real number* $\varepsilon > 0$ *there exists a real number* $\gamma > 0$ *such that for any interval* $A \in I(E)$ *with* $\vartheta(A) < \gamma$ *we have*

$$\|G(A)\| < \varepsilon.$$

2.4 Measurable Function

Let X be a locally compact nondiscrete metric space X, μ^* be an outer measure generated by a function volume ϑ on $E(X)$, $E \subset X$, and $(Y, \|.\|)$ be a norm space with θ be a null vector of Y and e be an unit vector (unit element) of Y. A function $\chi_E: X \to Y$ is said to be a **characteristic function** on E if

$$\chi_E = \begin{cases} e \ if \ x \in E \\ \theta \ if \ x \notin E \end{cases}$$

([7], p. 656-662).

A function $s: X \to Y$ is said to be a **simple function** if there are a finite family of disjoint μ^* −measurable sets $\{X_1, X_2, X_3, \dots, X_n\}$ and finite real numbers $c_1, c_2, c_3, \dots, c_n$ such that

$$X = \bigcup_{i=1}^{n} X_i$$

and

$$s(x) = \sum_{i=1}^{n} c_i \chi_{X_i}(x).$$

If c_i are distinct, then this representation is called canonical representation of s.

A function $g: X \to Y$ is said to be μ^* −measurable on μ^* −measurable set E if there is a sequence of simple function (s_n) on E such that (s_n) converge to g allmost everywhere on E ([7], p. 656-662).

Theorem 2.4.1. ([7], p. 656-662). *If a function* $g: X \to Y$ *is Henstock* ϑ *−integrable* on a *cell* $E \subset X$*, then* g *is* μ^* −measurable on E.

3. Result and Discussion

In this section, the locally compact nondiscrete metric space denoted by X and norm space Y by $(Y, \|.\|)$.

Based on the concept of locally small Riemann sum (LSRS) property of a al valued function on a cell $[a,b] \subset \mathbb{R}$, we construct LSRS property of a vector valued function on a cell $E \subset X$ with respect to volume function ϑ.

Definition 3.1 *Let E be a cell on X and ϑ be a volume function on $E(X)$. A function $g: X \to Y$ is said to have locally small Riemann sum property with respect to ϑ on E, and denoted by $g \in LSRS(E,\vartheta)$ if g is μ^* $-$measurable on E and for any real number $\varepsilon > 0$ there exists a function $\delta: E \to \mathbb{R}^+$ such that for each $t \in E$ and for any Perron $\delta -$ fine partition $P = \{(A_x, x)\}$ of cell $F \subset N(t,\delta(t)) \cap E$ we have*

$$\left\| P \sum g(x)\vartheta(A_x) \right\| < \varepsilon.$$

will prove that the $LSRS(E,\vartheta)$ is linear space, i.e. if $f, g \in LSRS(E,\vartheta)$, then $f + g \in LSRS(E,\vartheta)$ and for any real number α we have $\alpha f \in LSRS(E,\vartheta)$.

Theorem 3.1. *If $f, g \in LSRS(E,\vartheta)$, then $f + g \in LSRS(E,\vartheta)$ and for any real number α we have $\alpha f \in LSRS(E,\vartheta)$.*

Proof: Let $\varepsilon > 0$ be real number. Since $f, g \in LSRS(E,\vartheta)$, there exists a function $\delta_1, \delta_2 : E \to \mathbb{R}^+$ such that for each $t \in E$ and any Perron $\delta_1 -$ fine partition $P = \{(A_x, x)\}$ of a cell $F \subset N(t,\delta_1(t)) \cap E$ and for $\delta_2 -$ fine partition $Q = \{(B_y, y)\}$ of a cell $F \subset N(t,\delta_2(t)) \cap E$ we have

$$\|P \sum g(x)\vartheta(A_x)\| < \frac{\varepsilon}{3|\alpha+1|} \quad \text{and} \quad \left\|Q \sum f(y)\vartheta(B_y)\right\| < \frac{\varepsilon}{3|\alpha+1|}.$$

Define a function $\delta: E \to \mathbb{R}^+$ with $\delta(x) = min\{\delta_1(x), \delta_2(x)\}$. If $t \in E$ and $T = \{(C_z, z)\}$ is Perron δ $-$fine of a cell $F \subset N(t,\delta(t)) \cap E$, then we have

$$\begin{aligned} \|T \sum (f+g)(z)\, \vartheta(A_z)\| &\leq \|T \sum f(z)\vartheta(A_z)\| + \|T \sum g(z)\vartheta(A_z)\| \\ &< \frac{\varepsilon}{3|\alpha+1|} + \frac{\varepsilon}{3|\alpha+1|} = \frac{2\varepsilon}{3|\alpha+1|} < \varepsilon. \end{aligned}$$

and

$$\begin{aligned} \|T \sum \alpha f(z)\, \vartheta(A_z)\| &= |\alpha| \|T \sum f(z)\vartheta(C_z)\| \\ &< \frac{|\alpha|\varepsilon}{3|\alpha+1|} < \varepsilon. \end{aligned}$$

So, $f + g \in LSRS(E,\vartheta)$ and $\alpha f \in LSRS(E,\vartheta)$. ■

Theorem 3.2. *If $g \in LSRS(E,\vartheta)$, then there exists a function $\delta: E \to \mathbb{R}^+$ such that*

$$\left\{P\sum g(y)\vartheta(B_y): P = \{B_y, y\}\ is\ a\ Perron\ \delta - fine\ partition\ of\ E\right\}$$
is bounded.

Proof: Let $\varepsilon > 0$ be a real number. Since $g \in LSRS(E,\vartheta)$, then there exists a function $\delta_1: E \to \mathbb{R}^+$ such that for each $t \in E$ and for any Perron $-\delta_1$ fine partition $P = \{(A_x, x)\}$ *of cell* $F \subset N(t, \delta_1(t)) \cap E$ we have

$$\|P\sum g(x)\vartheta(A_x)\| < \varepsilon \qquad (1)$$

Really we have

$$\{N(x, \delta_1(x): x \in E)\} \qquad (2)$$

is an open cover of a cell E. Since E is a compact set, then there exists a finite subset of (2),

$$\{N(x_i, \delta_1(x_i)); x_i \in E, i = 1,2,3, \dots, n\} \qquad (3)$$

such that (3) is open cover of E. So, for each $x \in E$ there exists

$$I_x \subset \{1,2,3, \dots, n\}$$

such that $x \in N(x_i, \delta_1(x_i)$ for any $i \in I_x$. So, we have

$$x \in \bigcap_{i \in I_x} N(x_i, \delta_1(x_i)).$$

Define a function $\delta: E \to \mathbb{R}^+$ with

$$\delta(x) = \frac{1}{3}\min\{d(x, \partial(N(x_i, \delta_1(x_i)): i \in I_x\}.$$

Let

$$P = \{(B_y, y)\} = \left\{\left(y_j, B_{y_j}\right): j = 1,2,3, \dots, m\right\}$$

be a Perron δ $-$fine partition on E. For any j there exists $i \in \{1,2,3, \dots, n\}$ such that $B_{y_j} \subset N(x_i, \delta_1(x_i)$ and $\left\{(B_{y_j}, y_j)\right\}$ is a Perron δ_1 $-$fine of a cell B_{y_j}. By using (1), we have

$$\left\|g(y_j)\vartheta(B_{y_j})\right\| < \varepsilon.$$

Finally, we have

$$\left\|P\sum g(y)\vartheta(B_y)\right\| \le \sum_{j=1}^{m}\left\|g(y_j)\vartheta\left(B_{y_j}\right)\right\| < m\varepsilon < \infty.$$

This means that

$$\left\{P\sum g(y)\vartheta(B_y): P = \{B_y, y\}\ is\ a\ Perron\ \delta - fine\ partition\ of\ E\right\}$$
is bounded. ■

We will prove the relations of $LSRS(E,\vartheta)$ and $H(E,\vartheta)$.

Theorem 3.3. *If* $g \in LSRS(E, \vartheta)$ *then for every cell* $I \subset in(E)$ *we have* $g \in H(I, \vartheta)$.

Proof: Let I be a cell subset of $int(E)$ and ε be a positive real number. Since $g \in LSRS(E, \vartheta)$, then there exists a function $\delta: E \to \mathbb{R}^+$ such that for each $t \in I$ and for any Perronδ −fine partition $P = \{(A_x, x)\}$ of a cell $F \subset N(t, \delta(t)) \cap$ I, we have

$$\left\| P \sum g(x)\vartheta(A_x) \right\| < \varepsilon.$$

This means that for any cell $F \subset N(t, \delta(t)) \cap I$ we have $g \in H(F, \vartheta)$. Since I is a cell, then I is a compact set. So, there are $t_1, t_2, t_3, \dots, t_n \in I$ and a finite collection of nonoverlapping cells

$$\{I_1, I_2, I_3, \dots, I_n\}$$

such that

$$I = \bigcup_{i=1}^{n} I_i$$

and for any $i \in \{1, 2, 3, \dots, n\}$, $I_i \subset N(t_i, \delta(t_i))$. So, $g \in H(I_i, \vartheta)$. Finally, we have

$$g \in H(\cup_{i=1}^{n} I_i, \vartheta) \quad \text{or} \quad g \in H(I, \vartheta). \quad \blacksquare$$

Theorem 3.4. *If* $g \in H(E, \vartheta)$ *then* $g \in LSRS(E, \vartheta)$.

Proof: Given a real number $\varepsilon > 0$. Since $g \in H(E, \vartheta)$, then by Theorem 2.4.1, g is μ^* −measurable on E. If G is primitive Henstock ϑ −integrable of a function g on E, then by Theorem 2.3.4, there exists a funtion $\delta_1: E \to \mathbb{R}^+$ such that for any Perron δ_1 −fine $P = \{(A_x, x)\}$ on E we have

$$P \sum \|g(x)\vartheta(A_x) - G(A_x)\| < \frac{\varepsilon}{3}. \quad (4)$$

Since G is primitive's Henstock ϑ −integrable of a function g on E, then by Theorem 2.3.5, there exists a real number $\gamma > 0$ such that for any cell $A \subset E$ with $\vartheta(A) < \gamma$, wehave

$$\|G(A)\| < \frac{\varepsilon}{3}. \quad (5)$$

Define two functions $\delta, \delta_2: E \to \mathbb{R}^+$ such that for each $x \in E$, we have

$$\vartheta(N(x, \delta_2(x))) < \gamma,$$

and

$$\delta(x) = min\{\delta_1(x), \delta_2(x)\}.$$

If $t \in E$ and $Q = \{(A_x, x)\}$ is a Perron δ −fine on a cell $F \subset N(t, \delta(t)) \cap E$, then by (4) and (5), we have

$$\left\|Q\sum g(x)\vartheta(A_x)\right\| \leq Q\sum\|g(x)\vartheta(A_x)-G(A_x)\|+\ Q\sum\|G(A_x\|$$
$$<\ \tfrac{\varepsilon}{3}+\tfrac{\varepsilon}{3}\ <\ \varepsilon.$$

This means that $g\in LSRS(E,\vartheta)$. ■

Theorem 3.5. *Given a function $g: E\ \to Y$. If $(Y,\|.\|)$ is a Banach space and $g\in LSRS(E,\vartheta)$, then $g\in H(E,\vartheta)$.*

Proof: Let ε be a positive real number. Since $g\in LSRS(E,\vartheta)$, then there exists a function $\delta: E\ \to\ \mathbb{R}^{+}$ such that for each $t\in E$ and for any Perron $\delta-$fine partition $P=\{(A_x,x)\}$ of cell $F\subset N(t,\delta(t))\cap E$ we have

$$\|P\sum g(x)\vartheta(A_x)\| < \tfrac{\varepsilon}{4}. \qquad (6)$$

Take any sequence of cells (E_n) with $E_n\subset in(E)$, $E_n\subset E_{n+1}$ for each $n\in\mathbb{N}$, and $\lim E_n=E$. By Theorem 2.3.1, for any $n\in\mathbb{N}$, we have $g\in H(E_n,\vartheta)$. So, for any $n\in\mathbb{N}$, there is a function $\delta_n: E_n\ \to\ \mathbb{R}^{+}$ such that for any partition $P_n=\{(A_x,x)\}$ on E_n, we have

$$\|(H-\vartheta)\int E_n\, d\vartheta-P_n\sum g(x)\vartheta(A_x)\|\ <\ \tfrac{\varepsilon}{3} \qquad (7)$$

Since $\lim E_n=E$, then there exist $K\in\mathbb{N}$ such that for any $n\in\mathbb{N}$ with $n\geq K$, we have

$$\vartheta(E-E_n) <\ \varepsilon.$$

For any $n,m\in\mathbb{N}$ with $n\geq K, m\geq K, n>m$, define a function $\delta: E\ \to\ \mathbb{R}^{+}$ sush as

$$\delta(x)=\begin{cases} min\{\delta_0(x),\delta_m(x),\delta_n(x),d(x,\partial(E_m)\}\ if\ x\in int(E_K)\\ min\{\delta_0(x),\delta_m(x),\delta_n(x)\} \quad if \quad x\in\partial(E_m)\\ min\{\delta_0(x),\delta_m(x),d(x,\partial(E_m))\}\ \ if \quad x\in E_n-E_m\end{cases}$$

Let

$$P_1=\ \{(A_x,x)\}=\ \{(A_1,x_1),(A_2,x_2),\dots,(A_l,x_l)\}$$

be a δ –fine partition on $cl(E_n-E_m)$. If $(A_x,x)\in P_1$, then by (6) we have

$$\|g(x)\vartheta(A_x)\|\ <\ \frac{\varepsilon}{4l}.$$

It consequence is

$$\|P_1\sum g(x)\vartheta(A_x)\|\ <\ \tfrac{\varepsilon}{4}. \qquad (8)$$

If $P_2\{(B_y,y)\}$ is aδ –fine partition on E_m, then $P_1\ \cup\ P_2$ is a δ –fine partition on E_n. By (7) and (8), we have

$$\left\| (H-\vartheta)\int_{E_m} g\, d\vartheta - (H-\vartheta)\int_{E_n} g\, d\vartheta \right\| \le \left\| (H-\vartheta)\int_{E_m} g - P_2 \sum g(y)\vartheta(B_y) \right\|$$
$$+ \left\| P_1 \sum g(x)\vartheta(A_x) + P_2 \sum g(y\, \vartheta(B_y) - (H-\vartheta)\int_{E_n} g\, d\vartheta \right\|$$
$$+ \| P_1 \sum g(x)\vartheta(A_x) \|$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

It means that $((H-\vartheta)\int_{E_n} g\, d\vartheta)$ is Cauchy sequece in Banach space $(Y, \|\ \ \|)$. So, $((H-\vartheta)\int_{E_n} g\, d\vartheta)$ convergent, and then $g \in H(E, \vartheta)$. ■

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